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Large amplitude surface waves

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A transformation technique is used to solve the problem of steady nonlinear surface waves where the restoring force is either gravity or surface tension. An exact nonlinear integro-differential equation is found which yields known approximate solutions. Extensions to the method to account for more complicated geometries are also illustrated. The equation is solved numerically and results in agreement with previous solutions are obtained. In the case of capillary waves, the existence of two types of wave of greatest height is clearly indicated.

1. Introduction

The problem of steady nonlinear two-dimensional waves has been studied extensively. An approximate solution for the case of gravity waves was obtained by Stokes (1847), and Michell (1893) investigated the wave of greatest height. Later, Nekrasov (1921) used a transformation to reduce the problem of gravity waves on a fluid of infinite depth to the solution of a nonlinear integral equation. This was solved numerically by Thomas (1968, 1975) and in fact an account of his method of solution and the nonlinear theory is given by Milne-Thomson (1968, p. 409). Also, reviews have been given by Wehausen & Laitone (1960) and Wehausen (1965). John (1953) solved the problem of a nonlinear wave on water of finite depth. However, the bottom surface was wavy but for sufficiently deep liquid, the amplitude of the bottom surface was negligible. A method used by Long (1956) was extended by Byatt-Smith (1969) to obtain an exact integral equation for the elevation of the free surface. This was used to derive known approximate solutions and to investigate solitary waves. More recently Schwartz (1974) extended Stokes' infinitesimal-wave expansion, using a computer to evaluate the coefficients. However, the expansion parameter used by Stokes limited the radius of convergence of the series to the extent that the highest wave was unobtainable. The domain of validity was extended through the use of Padé approximants and estimates of the highest wave were obtained. A similar method, using Padé approximants to extend the range of convergence, was employed by Longuet-Higgins (1975) to examine the behaviour of large amplitude waves in deep water. These computations suggested that the speed of the waves was greatest when the wave height was just less than the maximum. This work has been extended by Cokelet (1977) to account for waves in water of finite depth. The work discussed so far has been concerned with gravity waves.

For the case of capillary waves on water of infinite depth, an exact solution has been found by Crapper (1957). He used the velocity potential and stream function as independent variables and solved the resulting equations of motion for the Cartesian coordinates x and y of the free surface. Very recently, Kinnersley (1976), following

Crapper's method, has obtained exact solutions for capillary waves on sheets of water of finite thickness.

In this paper, a transformation method is used to solve the problem of steady nonlinear waves where the restoring force is either gravity or surface tension. A transformation is found which maps the region occupied by the fluid into a half-plane. An exact nonlinear integro-differential equation is then obtained for the slope of the free surface. Although, for the present, attention is confined to the case of periodic waves over a flat bottom, the mapping could be modified so that neither of these restrictions need be imposed. However, the resulting difficulties would be beyond the scope of the present paper.

The integro-differential equation is expanded in terms of the amplitude and solved to give known results such as the linear theory and Stokes solution. For the case of pure capillary waves Crapper's exact solution is expanded in the appropriate manner and is shown to agree with the approximate solution obtained from the present theory. For more general cases a truncated Fourier series is used to represent the slope of the free surface. The coefficients in the series are determined by substituting the expression into the integro-differential equation at certain points and solving numerically the resulting set of nonlinear algebraic equations.

For the case of gravity waves, the results are in agreement with solutions previously obtained. Although the solitary wave cannot be dealt with directly by the present method, the way in which the solution approaches this particular wave is examined in the results. In dealing with capillary waves on a liquid of finite depth, the results indicate two types of limiting form of the wave of largest amplitude for a given mean depth. The relevance of this phenomenon in the breakup of thin films is mentioned.

2. The wave problem

The problem considered is that of steady two-dimensional waves on the free surface of a liquid flowing over a flat horizontal bottom. A co-ordinate system is chosen in which the wave is stationary, the x axis is horizontal and the y axis points vertically upwards.

Assuming that the flow may be regarded as inviscid and irrotational, the problem is reduced to finding a complex potential satisfying the appropriate boundary conditions. This can most conveniently be done by mapping the region occupied by the fluid into a somewhat simpler region, the lower half of the ζ plane say. The boundary of the z plane, which is inclined at an angle α to the real axis, is mapped into the real axis of the ζ plane by the transformation

$$\frac{dz}{d\zeta} = C \exp \int_{-\infty}^{\infty} \frac{1}{\pi} \frac{\alpha(t)}{\zeta - t}\, dt \tag{1}$$

since $dz/d\zeta$ is an analytic function whose natural logarithm has α as its imaginary part on the real axis. Furthermore, by making the origin in the ζ plane correspond to $-\infty$ in the z plane and the $-\xi$ axis ($\zeta = \xi + i\eta$) correspond to a flat bottom in the physical plane, as shown in figure 1, this transformation may be simplified to give

$$\frac{dz}{d\zeta} = \frac{C}{\zeta} \exp \int_{0}^{\infty} \frac{1}{\pi} \frac{\alpha(t)}{\zeta - t}\, dt. \tag{2}$$

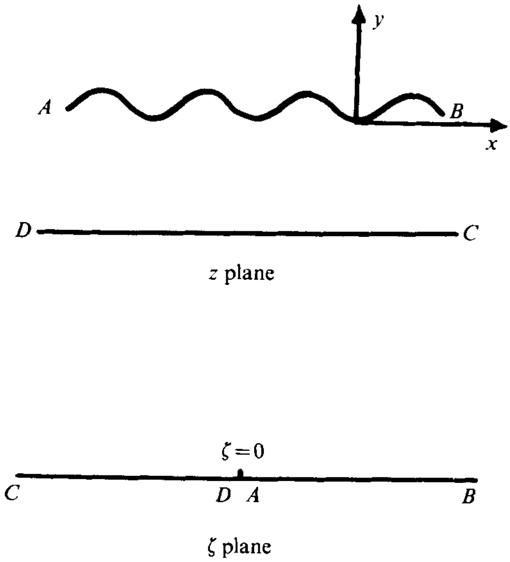

FIGURE 1. The correspondence between the z plane and the ζ plane under the transformation (3).

The velocity components in the x and y directions respectively are u and v. Since $u-iv$ is a function analytic in the lower half of the ζ plane with $-\alpha$ as the imaginary part of its natural logarithm on the boundary,

$$u-iv = \frac{Q}{\pi C}\exp\int_0^\infty -\frac{1}{\pi}\frac{\alpha(t)}{\zeta-t}dt. \tag{3}$$

Here Q represents the flux of fluid in the $+x$ direction in the z plane, since the complex potential in the ζ plane is that associated with a line source of strength Q/π situated at the origin.

The condition to be satisfied at the free surface is simply Bernoulli's equation. Allowing for the effects of both gravity and surface tension, this is

$$\tfrac{1}{2}\rho(u^2+v^2)+\rho g y_s - T\cos\alpha\, d\alpha/dx = \text{constant} \tag{4}$$

if the pressure over the surface is taken as uniform. The quantities ρ, g and T denote density, acceleration due to gravity and the surface-tension coefficient respectively. The terms involving α represent the curvature of the free surface. In general, the constant on the right-hand side of (4) will depend on the mass transport induced in the fluid by the waves and therefore it is found more convenient to use the derivative with respect to $\ln\xi$ of this equation.

From (3), at the free surface the fluid speed is given by

$$|u+iv| = \frac{Q}{\pi C}\exp\left\{-\mathrm{P}\int_0^\infty \frac{1}{\pi}\frac{\alpha(t)}{\xi-t}dt\right\},$$

where $\mathrm{P}\int$ denotes the principal value of the integral. Using this, and the derivative of (4), the free-surface condition becomes a nonlinear integro-differential equation for α, namely

$$-\frac{Q^2}{\pi^2C^2}\frac{dP}{dr}+gCe^{3P}\sin\alpha+\frac{T}{\rho C}e^P\left(\frac{d\alpha}{dr}\frac{dP}{dr}-\frac{d^2\alpha}{dr^2}\right) = 0, \tag{5}$$

where

$$P = \mathrm{P}\int_0^\infty \frac{1}{\pi}\frac{\alpha(t)}{e^r - t}dt, \quad r = \ln\xi.$$

This is an exact formulation of a free-surface problem for flow over a flat horizontal bottom. By referring to (1), it can be readily appreciated that the modifications to (5) arising from a change in the geometry are easily made. However, discussion in this paper is concerned only with steady periodic waves. These are obtained when $\alpha(\xi)$ is a periodic function of $\ln\xi$ and under these circumstances the integral in the transformation converges only in the sense of a generalized function.

3. Approximate solutions

A solution to (5) may be obtained when α is small. If terms involving products in α are neglected, the equation reduces to

$$-\frac{Q^2}{\pi^2C^2}\frac{dP}{dr} + gC\alpha - \frac{T}{\rho C}\frac{d^2\alpha}{dr^2} = 0. \tag{6}$$

Since

$$\mathrm{P}\int_0^\infty \frac{1}{\pi}\frac{\sin(k\ln t)}{e^r - t}dt = -\frac{\cos kr}{\tanh k\pi}, \tag{7}$$

it is easily seen that the solution of (6) is

$$\alpha = a\sin kr,$$

where

$$-\frac{Q^2k}{\pi^2C^2\tanh k\pi} + gC + \frac{Tk^2}{\rho C} = 0 \tag{8}$$

and a is an arbitrary small constant. Using (2) with this form of α and making approximations consistent with those already made, it can be shown that

$$y_s = (-Ca/k)\cos kr, \quad x_s = Cr$$

to within arbitrary additive constants. The mean depth of the fluid is $C\pi$ ($= h$ say). Hence

$$y_s = (-a/K)\cos Kx_s,$$

and using (8) gives

$$U^2 = \left(\frac{g}{K} + \frac{T}{\rho}K\right)\tanh Kh,$$

where $K = k/C$ is the wavenumber and $Q = Uh$. Since the mass transport induced by the waves is negligible to this order of approximation, the mean stream speed U is the phase speed of the waves. So it is seen that the familiar linear wave solution is recovered from this analysis.

Higher-order solutions of (5) could be obtained by expanding the equation in powers of α. However, the nonlinear terms in α generate higher harmonics and it will be demonstrated that (5) can be solved more simply using an expansion for α which involves the higher harmonics and is in terms of a small parameter related to the amplitude of the wave. The order of magnitude of the coefficient of a harmonic in this expansion decreases as the order of the harmonic increases.

For the moment, discussion is restricted to the case of gravity waves, so that the free-surface condition (5) becomes

$$-\frac{Q^2}{\pi^2 C^2}\frac{dP}{dr}+gCe^{3P}\sin\alpha = 0. \tag{9}$$

Stokes' (1847) solution for the case of waves on water of infinite depth can be obtained by taking the first three terms in an expansion for α, i.e.

$$\alpha = a\sin kr+\beta a^2\sin 2kr+\gamma a^3\sin 3kr, \tag{10}$$

where a is a small parameter, in fact proportional to the amplitude of the waves. The constants β and γ are independent of a and may be regarded as being of order unity.

The expression (10) for α allows the value of P to be deduced from (7). However, the case of fluid of infinite depth is obtained by allowing $k\to\infty$ and hence $\tanh k\pi = 1$. The algebra is simplified if this approximation is made at this early stage, so that P is being given by

$$P = -a\cos kr-\beta a^2\cos 2kr-\gamma a^3\cos 3kr.$$

Substituting in (9) for α and P and neglecting terms $O(a^4)$, it can be shown that

$$Q^2k/\pi^2C^3g = 1+a^2, \quad \beta = -\tfrac{3}{2}, \quad \gamma = \tfrac{17}{6}. \tag{11}$$

By using these results for α in the transformation (2), an equation for the free surface is determined in terms of the parameter r:

$$x = C\left(r-\frac{a}{k}\sin kr+\frac{a^2}{k}\sin 2kr\right)$$

and

$$y = C[(a+\tfrac{1}{8}a^3)\sin kr-2a^2\sin 2kr+\tfrac{9}{2}a^3\sin 3kr].$$

After some algebra to eliminate r, the equation for the free surface becomes

$$y = -A\cos Kx+\tfrac{1}{2}KA^2\cos 2Kx-\tfrac{3}{8}K^2A^3\cos 3Kx+\tfrac{1}{2}KA^2,$$

where $K = k/C$ is the wavenumber and

$$A = (C/k)(a+\tfrac{5}{4}a^3). \tag{12}$$

To determine the phase speed, it is observed that, if the complex potential w is expressed in the form $w = \Phi+i\psi$, then the change in Φ over a wavelength is $U\lambda$, where U is the phase speed of the waves and λ is the wavelength. On the flat bottom, for the case of liquid of finite depth, or at depths large compared with a wavelength, for infinite depth, the velocity component v is zero, so that $u = dw/dz$. Hence U is given by

$$U\lambda = \int_0^\lambda \frac{dw}{dz}\bigg|_D dx = \int_{x=0}^{x=\lambda}\frac{Q}{\pi\zeta}d\zeta = \frac{2Q}{k}, \tag{13}$$

where D is a 'sufficiently large depth', since the change in $\ln\zeta$ over a wavelength is $2\pi/k$.

The condition expressed by (11), using (12) and (13), becomes

$$U^2K/g = 1+A^2.$$

These results are in agreement with the expressions obtained by Stokes (see, for example, Kinsman 1965, p. 251).

The case of nonlinear capillary waves is now considered. The surface condition (5) for this situation becomes

$$-\frac{Q^2}{\pi^2 C^2}\frac{dP}{dr}+\frac{T}{\rho C}e^P\left(\frac{d\alpha}{dr}\frac{dP}{dr}-\frac{d^2\alpha}{dr^2}\right)=0. \tag{14}$$

As before, an approximate solution can be found with α expressed by (10). The restriction to infinite depth is again imposed in order that comparisons may be made with the exact solution obtained by Crapper (1957).

Using (14) and following the procedure adopted for the case of gravity waves, the coefficients in the expansion for α can be found together with a condition on the phase speed of the waves:

$$\beta = 0, \quad \gamma = \tfrac{1}{48}$$

and

$$\pi^2 CkT/Q^2 = 1+\tfrac{1}{8}a^2. \tag{15}$$

Using these results and (2), the form of the free surface can be found in terms of the parameter r: as

$$\left.\begin{aligned} x_s &= Cr-\frac{Ca}{k}\sin kr+\frac{Ca^2}{4k}\sin 2kr \\ \text{and}\quad y_s &= -\frac{C}{k}\left(a\cos kr-\tfrac{1}{4}a^2\cos 2kr+\tfrac{1}{16}a^3\cos 3kr\right)\end{aligned}\right\} \tag{16}$$

to within arbitrary additive constants of integration. As before, the wavenumber $K = k/C$. Rather than eliminating r, it is convenient to compare the solution with Crapper's results in this form.

The exact solution for the form of the free surface in terms of a parameter s is given by

$$\frac{x-iy_s}{\lambda}=s-\frac{2i}{\pi}\left[1+\frac{2\lambda}{\pi A}\left\{\left(1+\frac{\pi^2A^2}{4\lambda^2}\right)^{\frac{1}{2}}-1\right\}e^{2\pi is}\right]^{-1}+\frac{2i}{\pi},$$

where A is the amplitude of the wave and λ is the wavelength. For small values of A this becomes

$$\frac{2\pi x}{\lambda}=2\pi s-\frac{\pi A}{\lambda}\sin 2\pi s+\frac{1}{4}\left(\frac{\pi A}{\lambda}\right)^2\sin 4\pi s\ldots,$$

$$\frac{2\pi y_s}{\lambda}=-\left[\frac{\pi A}{\lambda}-\frac{1}{16}\left(\frac{\pi A}{\lambda}\right)^3\right]\cos 2\pi s+\frac{1}{4}\left(\frac{\pi A}{\lambda}\right)^2\cos 4\pi s-\frac{1}{16}\left(\frac{\pi A}{\lambda}\right)^3\cos 6\pi s\ldots.$$

Clearly these are the same as expressions (16) to the required order if we put $2\pi s = kr$ and $\pi A/\lambda-\frac{1}{16}(\pi A/\lambda)^3 = a$. The phase speed U is determined from (15) by using (13), and is given by

$$U^2=\frac{TK}{\rho}\left[1-\frac{1}{8}\left(\frac{\pi A}{\lambda}\right)^2\right]+O(A^4).$$

This agrees with the exact result

$$U^2=\frac{TK}{\rho}\left(1+\frac{\pi^2A^2}{4\lambda^2}\right)^{-\frac{1}{4}}.$$

4. Numerical solution

To find a numerical solution of (9) or (14), α is expressed in the form of a truncated Fourier series, i.e.

$$\alpha = \sum_{n=1}^{N} a_n \sin nkr.$$

To determine the N unknown coefficients $a_1, a_2, \ldots, a_N$ this expression for α is made to satisfy (9) or (14) at N interior points, for example $kr = i\pi/(N+1)$, where $i = 1, 2, \ldots, N$. Clearly the end points are satisfied by the form chosen for α. As a result of this substitution, N simultaneous nonlinear algebraic equations are obtained for the unknown a_n's.

For the case of gravity waves, a parameter $\mu = Q^2k/\pi^2gC^3$ is introduced and clearly $\mu = 1$ for infinitesimal waves on water of infinite depth. By a minimization technique the algebraic equations resulting from (9) are solved numerically for various values of the parameters μ and k. The numerical iterative procedure can be started by using the previously obtained third-order solution as a first guess when μ is close to unity. For other values of μ at a fixed value of k the initial guesses are obtained from previous solutions.

The problem of capillary waves is dealt with in a similar manner except that the relevant parameter is $\nu = Q^2\rho/\pi^2CkT$, with $\nu = 1$ again corresponding to infinitesimal waves on deep water.

With the values of the a_n's known, (2) can be integrated to give the shape of the wave and thus the wavelength. From (13) the phase speed of the waves can be found. For the case of gravity waves U is given by

$$\frac{U^2}{g\lambda/2\pi} = \left(\frac{2C\pi}{k\lambda}\right)^3 \mu,$$

while for capillary waves the relationship is

$$\frac{U^2}{2\pi T/\rho\lambda} = \frac{2\pi C}{k\lambda}\nu.$$

If $\zeta = e^{i\theta}$ is substituted into the transformation (2), the depth d of water below a wave trough can be found by integrating the equation over $-\pi \leqslant \theta \leqslant 0$. That is,

$$d = C\int_{-\pi}^{0} \exp\left\{-\sum_n a_n \frac{\cosh k(\pi+\theta)}{\sinh k\pi}\right\} d\theta, \tag{17}$$

where the terms under the summation have been obtained by evaluating the integral exponent in (2) around a suitably deformed contour. Hence, since the wave form has been determined, the mean depth h of water can be found after evaluating the integral in (17).

5. Results

Concentrating first of all on the case of gravity waves, the value of N was initially chosen to be 10. The values of kr substituted in (9) to obtain the nonlinear algebraic equations were equally spaced in the interval $[0, \pi]$.

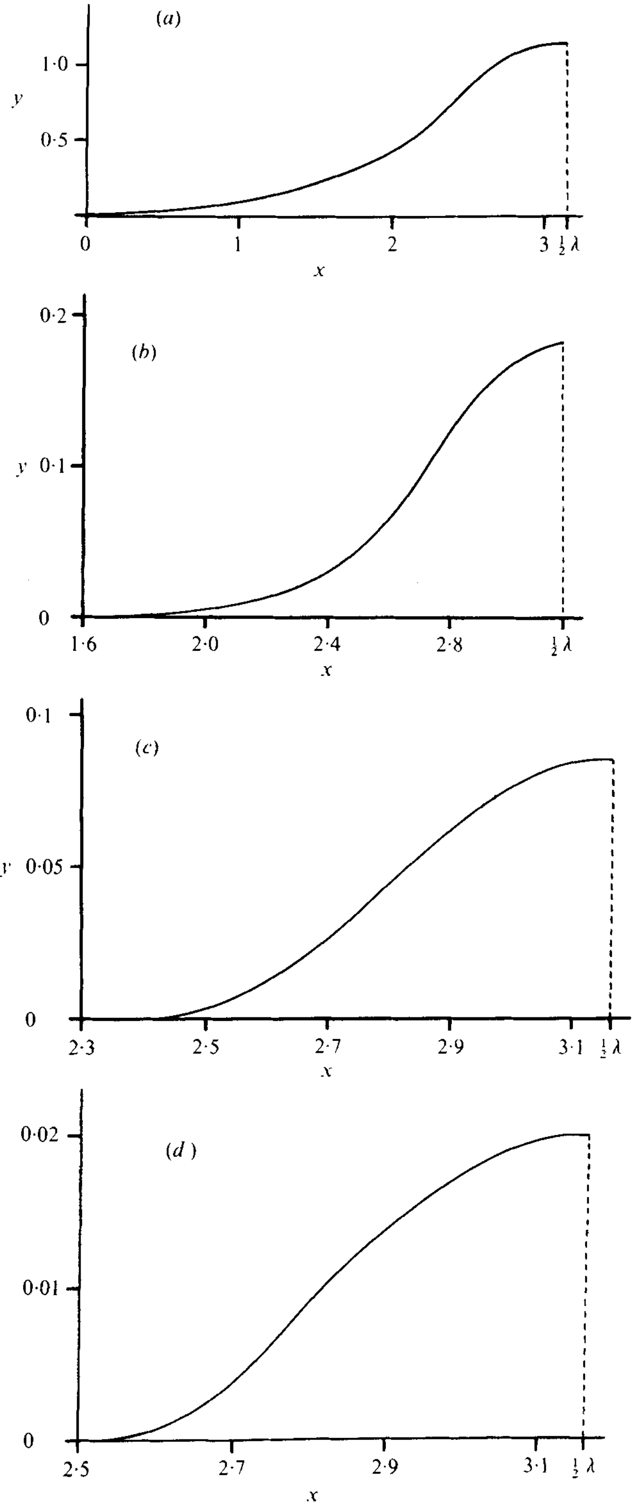

FIGURE 2. The wave forms for various values of h/λ and A/λ, showing the approach to a solitary wave. Note that the free surface is approximately flat from $x = 0$ up to the first value of x shown in (*b*)–(*d*). (*a*) $h/\lambda = 0{\cdot}57$, $A/\lambda = 0{\cdot}187$. (*b*) $h/\lambda = 4{\cdot}6 \times 10^{-2}$, $A/\lambda = 2{\cdot}87 \times 10^{-2}$. (*c*) $h/\lambda = 2 \times 10^{-2}$, $A/\lambda = 1{\cdot}36 \times 10^{-2}$. (*d*) $h/\lambda = 5{\cdot}1 \times 10^{-3}$, $A/\lambda = 3{\cdot}14 \times 10^{-3}$.

In figure 2, the wave forms for various values of the parameters are shown. Figure 2(*a*) shows the typical shape of a nonlinear gravity wave on water of moderate depth, i.e. a sharpened crest and flattened trough. As the values of A/λ and h/λ are reduced the flat trough is seen to extend over a still larger region and the wave speed relative to that of an infinitesimal long wave increases. This tendency is illustrated in figures 2(*b*)–(*d*). In figure 2(*d*), where h/λ has been reduced to 5×10^{-3} while A/h is very close to the maximum permissible value, the trough is quite flat for over 80 % of the wavelength and a solitary wave is approached.

In the numerical results, there were slight wobbles, of amplitude less than 1 % of the wave amplitude, which arose from the inaccuracies implicit in the numerical scheme. Since the derivative of the Bernoulli free-surface condition is used in the solution of the algebraic equations, it is possible to carry out a direct check on the accuracy of the results. By substituting the computed values of the wave height and fluid speed into Bernoulli's equation applied at the free surface, the errors in the solutions can be estimated.

With the ten terms taken, and for waves near their maximum amplitude, the errors in some cases were almost 10 %. By increasing the value of N to 20 but keeping the points at which (9) was satisfied equally spaced, this error was reduced to less than 3 %. However, the computing time required to obtain convergence to a solution of the required accuracy increased by a factor of three. This could no doubt be improved by choosing the N points at which (9) is satisfied more carefully, paying particular attention to regions of rapid variation. For the present results this was not done as the desired accuracy was achieved for very small expenditure in computing time.

In figure 3 the variation of the wave speed with amplitude is shown. The maximum value of A/h is taken from Wehausen & Laitone. In terms of the present analysis, the wave of greatest height for a given μ would be reached when the value of k was just small enough to make to the principal value P tend to $-\infty$. This can be thought of as the minimum value of k for which the series

$$\sum_{n=1}^{\infty} a_n \frac{(-1)^n}{\tanh nk\pi}$$

converges [see (7)]. Since the series is truncated in the present calculations, the wave of maximum height cannot be determined. A similar condition arose in the Schwartz analysis and use of a Domb–Sykes plot allowed the maximum value of the amplitude to be estimated. This approach is inappropriate in the present analysis because no formal expansion procedure is used, rather the surface condition is satisfied at discrete points.

Comparisons with the direct numerical calculations of Chappelear were made and showed excellent agreement. This makes the present method particularly useful in the region where the higher-order Stokes solutions are known to be in error, i.e. as h/λ decreases, so that U^2/gh becomes greater than unity. However, the calculations of Schwartz, where the range of convergence of the Stokes solution was extended by the use of Padé approximants, were thought to predict the Froude number to an accuracy of 1 % for waves up to the highest and for h/λ as small as 0·0570. A comparison of the present calculations with these results for $h/\lambda = 0{\cdot}035$ and a/h in the range for which Schwartz's results are accurate is shown in figure 3 and it can be seen that extraordinarily good agreement is obtained. The curve for the solitary wave, also taken from

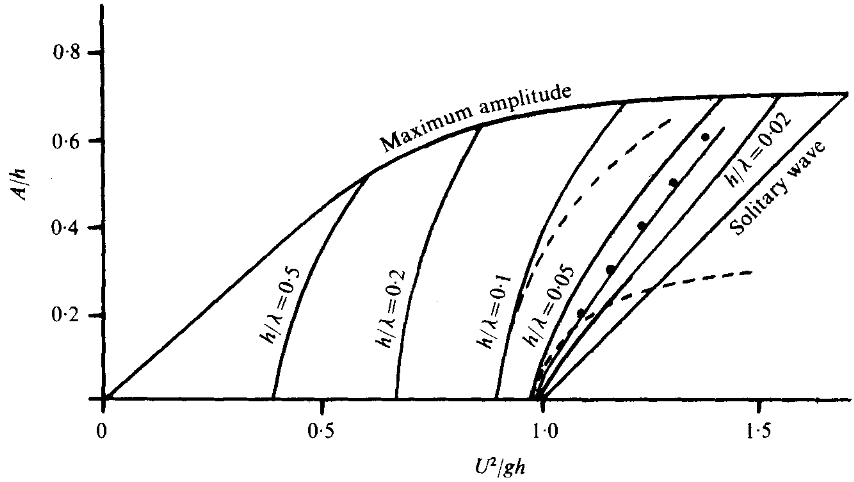

FIGURE 3. The variation of wave speed with amplitude for gravity waves. ---, Stokes' fifth-order solution, which for $h/\lambda = 0{\cdot}2$ is virtually identical to the present calculations; ●, from Schwartz's solution for $h/\lambda = 0{\cdot}0356$.

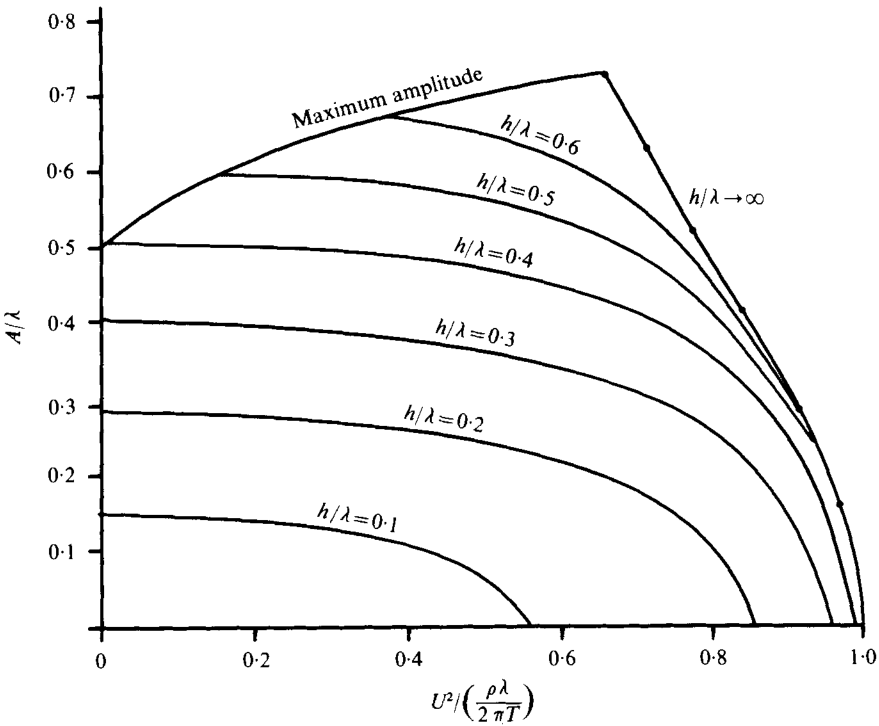

FIGURE 4. The variation of wave speed with amplitude for capillary waves. ●, particular examples of Crapper's exact solution for infinite depth.

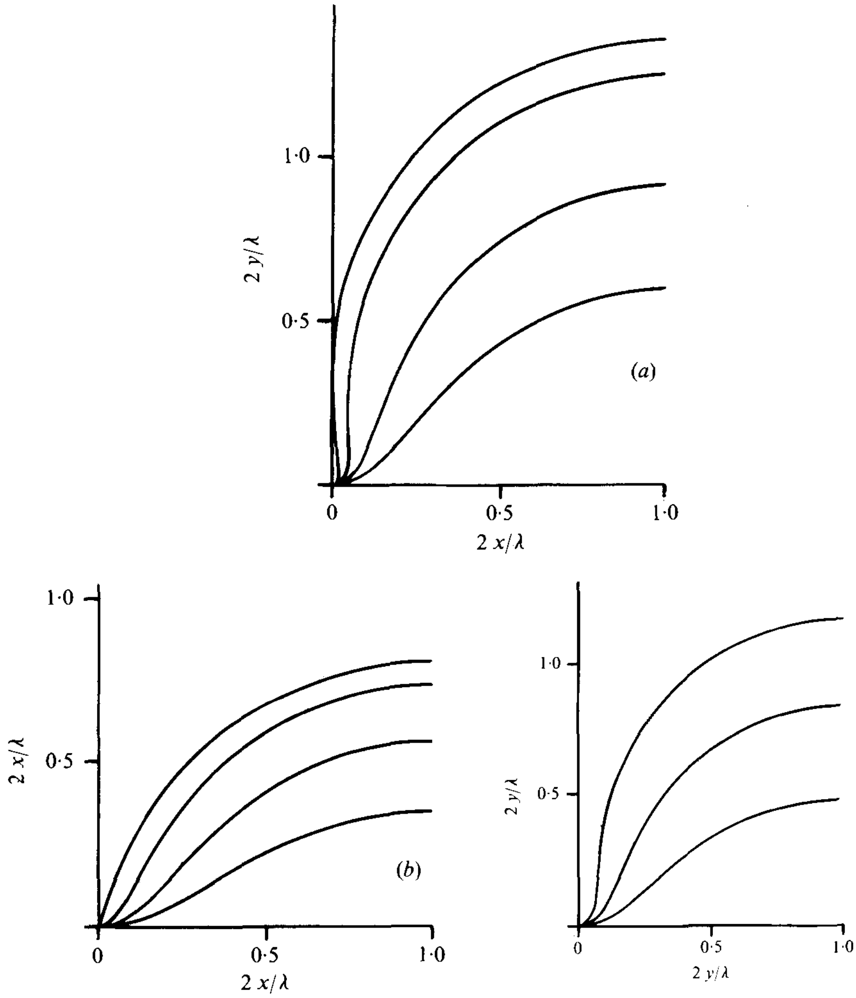

FIGURE 5. Capillary wave profiles for various values of h/λ. (a) $h/\lambda = 0{\cdot}6$. (b) $h/\lambda = 0.3$. (c) $h/\lambda \rightarrow \infty$.

Wehausen & Laitone, is shown to indicate how close to this limiting solution the present approach has been carried. No attempt has been made to perform the calculations to the accuracy required to corroborate the recent findings of Cokelet, with regard to the fastest-moving wave. However, such calculations should be possible even though the wave height will not be a monotonically increasing function of μ, since μ is not used as an expansion parameter. For a given value of μ, there is no reason why, in principle, two physically realistic solutions to the set of algebraic equations cannot be found. However, in practice the effort involved in attaining the required accuracy would be enormous.

Coming now to the case of capillary waves, figure 4 shows the variation of wave speed with amplitude for various values of h/λ. Also shown in the figure is the exact result of Crapper, for waves on water of infinite depth, which is compared with the present calculations. As can be seen, the results are indistinguishable. The general trend of increasing amplitude leading to decreasing wave speed is shown and also for a given wave speed the amplitude increases rapidly with the mean depth.

The curve referring to the waves of maximum amplitude is obtained by calculating the wave which just touches its neighbour, enclosing a pocket of air. This curve cuts the A/λ axis at about 0·5 and thus two distinct types of wave are represented in the figure. The curves h/λ = constant which cut the maximum amplitude curve have their height restricted in the way already described. The wave represented by the curves h/λ = constant which cut the A/λ axis are restricted in amplitude in an entirely different way.

Figures 5(*a*) and (*b*) illustrate this point. In figure 5(*a*), $h/\lambda = 0{\cdot}6$ and the wave profiles at various speeds are shown, including the wave of maximum amplitude for this value of h/λ. When $h/\lambda = 0{\cdot}3$, the wave profiles are as shown in figure 5(*b*), and the significant feature of these waves is the depth of liquid below the wave trough. This depth, made non-dimensional by the wavelength, is approximately 0·0003 for the largest wave shown and increases through the values 0·045 and 0·12 to 0·2 for the smallest wave. Evidently this type of wave is restricted in amplitude by the amount of liquid available and is clearly of importance in the symmetrical breakup of thin liquid films.

These findings are in agreement with the results of Kinnersley (1976), who considered two distinct types of wave, the limiting form separating the two types having an elliptic profile. Figure 5(*c*) shows the wave profiles for large values of h/λ, which are of the form obtained by Crapper.

6. Conclusions

It has been shown that the method developed in this paper is capable of reproducing known approximate solutions to problems in nonlinear water waves. The method of solution of the exact equation for the free-surface slope α is numerically simple and efficient with typical computing times of about 10 s on the 1906A at Leeds and, for both gravity waves and capillary waves, the results are in excellent agreement with previous calculations. Furthermore, in the case of capillary waves on water of finite depth a whole range of information is obtained and the two types of wave of greatest height examined.

Previous work on developing an exact equation for nonlinear free-surface waves using a mapping technique, as originally carried out by Nekrasov, formulated a periodic wave. With this constraint, a region between the bottom and the free surface and bounded by vertical planes a wavelength apart was mapped into a circular annulus, the mapping being given by an infinite series. Use of Levi-Civita's (1925) surface conditions then allowed the exact equation for the free surface to be found.

In the present method, the whole of the region occupied by the fluid is mapped into the lower half of the ζ plane, where a complex potential may be written down. Hence the solution for periodic waves is a particular case of a more general approach to nonlinear free-surface problems. As indicated earlier, the method gives an exact equation for the free surface for flow over submerged obstacles and could even deal with partially immersed bodies on the surface.

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